Article

# The Art of Generalization

by Arpit Dwivedi

The aim of this note is to look at some famous generalizations done in the history of mathematics and to have an opportunity to see various ideas and motivations behind the generalizations. This note is prepared with a general reader in mind therefore, I don't intend to go deep with the ideas, rather I'll give various examples to illustrate the ideas behind those generalizations.

## Introduction

In mathematics, the term generalization refers to extending a concept, idea, principle, or mathematical object (such as numbers or functions) by changing some of the initial assumptions to a broader area of applicability. For example, if we look at the expression $y = mx$, at first, it may seem the family of straight lines passing through the origin. But in three dimensions, it represents a plane; moreover, if we relax the untold, underlying assumption that- the expression has only two variables and generalize it to $n$ variables, we see that it represents a hyper-surface of dimension $(n-1)$ sitting in an $n$-dimensional space.

This generalization allows us to unite the idea of a family of hyper-surfaces into a single representation, which makes the study of higher dimensions one step closer to comprehend. This is exactly the purpose of generalization, that is to get our initial idea to work on a broader domain so that we can have better tools to understand the world around us.

The aim of this article by no means is to give all the generalizations that have ever occurred, nor it is to teach how to generalize rather, it is to motivate the readers to dive deep into the sea of mathematics by introducing some famous, useful, and elegant examples. To start things off, let me motivate you with arguably the most important generalization ever done in Geometry. That is relaxing one of the Euclid's postulates[1] (which gave us the Euclidean Geometry) to open the gateways to very counterintuitive non-Euclidean Geometry, such as spherical Geometry and hyperbolic Geometry. With these geometries, today we can talk about the shape of the Universe.

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[1] Heath, Thomas L. *The Thirteen Books of Euclid's Elements*, Dover Publications, Inc, 1956.

## Examples

Let us now discuss a few fundamental and insightful examples which have occurred throughout the history of mathematics.

### The Number System

We started our journey of numbers with natural numbers used as counting numbers and soon realized that something was missing. Suppose a person A has two friends B and C whose homes are in the same street as that of A with A's home in the middle. Both B and C's homes are 25 steps away from A's home. If A has gone 25 steps from his home, we don't know whether he went to B's home or C's home. Hence, what's missing with the information '25 steps' is the sense of direction. To remove this obstacle, the integers were discovered.

Similar necessities were observed and helped to improve the number system further. For example, dividing a cake equally for a number of persons and then asking how much of the cake a person got- created a new type of number, called the rational numbers.

Squaring a number is easy but with the information of rational numbers, the question 'What happens when we take the square root of a number which is not a perfect square?' leads to the real numbers and similarly, taking the square root of a negative number was also absurd, until we came up with the idea of complex numbers.

### The Gamma Function

We all know what a factorial is. $n! = 1 \cdot 2 \cdot 3 \cdots n$, where $n$ is a natural number. But what if $n$ is not a natural number? This time, it was not at all obvious how to answer this, and the question had to wait till 1729 for Euler, when he noticed that $n!$ has an integral representation as the following:

$$n! = \int_0^\infty e^{-t} t^n dt.$$

Euler again noticed that this integral makes sense not only for natural numbers, but also for whole positive reals, and it also lets you define the zero factorial. With some modifications, Euler was able to define the Gamma function, which is seen as the generalization of the factorial function, for the whole complex plane, except for negative integers and zero, where it has simple poles. The final definition is as follows:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad z \in \mathbb{C}.$$

More about the Gamma function can be found in the book[2].

[2] Artin, Emil. *The gamma function*, Courier Dover Publications, 2015.

### From Newton to Einstein

Let me ask you some questions. What is Time? What is Gravity?

$\vdots$

Felt a shortage of words, right? In search of answers to these questions (especially, the gravity one) Newton and Einstein both gave their theory: Newton's laws of motion and Einstein's Special and General Theory of Relativity[3].

Newton's laws fail at instances where very high speeds (a significant portion of the speed of light) are involved, and if the object under study is under the influence of a very massive celestial body (like a black hole or a massive star). While Einstein's equations work perfectly fine in these situations.

Einstein's Theory of Special Relativity is the consequence of an attempt to answer a very simple question: 'Am I in motion?', under the condition that no acceleration is involved. When acceleration is involved, the Special Theory is replaced by the General Theory of Relativity. The fascinating thing about the General Theory is that- if you put acceleration as zero, you will get the Special Theory of Relativity, and if you apply low speeds in the Special Theory of Relativity, you will get Newton's laws.

### The Fractional Derivatives

One of the hot topics today for researchers is the fractional derivative. Newton and Leibniz both are credited for the invention of the classical derivatives, but it was Leibniz who used the symbol $\dfrac{d^n}{dx^n}$ to denote the $n$-th derivative with respect to the independent variable $x$. This symbol was immediately accepted, as it was better than Newton's symbols (which were dashes) in the sense that it also tells you what the independent variable is. L'Hôpital, after gazing at his symbol, wrote a letter to Leibniz asking what happens if $n = 1/2$. This was the moment when fractional derivatives were born[4].

Evidently, Leibniz did not have a satisfactory answer for that and said, *someday very useful consequences will be drawn from this apparent paradox.* The question was ahead of its time because the modern definition of fractional derivative includes the Gamma function, which was discovered in 1729 by Euler. Today, we are able to define fractional integrals and derivatives precisely and use them in very wide areas of mathematics. With computers in our grasp, these derivatives are proving to be a better mathematical tool for modeling physical phenomena.

[3]Miller, Arthur I. *Albert Einstein's Special Theory of Relativity: emergence (1905) and early interpretation (1905-1911)*, Reading, MA: Addison-Wesley, 1981.

[4]Miller, K. and Ross, B., *An Introduction to the Fractional Calculus and Differential Equations*, Wiley, New York, 1993.

There are many ways to reach fractional derivatives from the classical one. Let me introduce here the easiest one (in my opinion). Observe that in the following formula for the multiple integrals (called the Cauchy-Euler formula for multiple integrals):

$$I^n f(x) = \frac{1}{\Gamma(n)} \int_{x_0}^{x} (x-t)^{n-1} f(t) dt,$$

the integral on the right-hand side makes sense even if $n$ is not an integer. It is allowed to be a real number and gives out an answer. This gives us an opportunity to define the fractional integrals, and then with the help of composition with integer order derivatives, we can define the fractional derivatives[5].

## Conclusion

Through these examples, we noticed that there is no straightforward recipe for generalization. The generalization can be done in different manners, in different situations like, (apart from the above examples,) we might do analytic continuation (which is a way to extend the domain of a function, such that it preserves some nice properties of the function) by power series method.

Obviously, there are many generalizations of the same thing therefore, we look for those generalizations that preserve some nice properties, and most importantly, it must return the same output as the original function, or object, when it is restricted to the condition of the function, or object, before generalization. For example, for positive integer values of $n$, the fractional derivative operator becomes the ordinary derivative operator.

[5]Podlubny, I., *Fractional Differential Equations, academic Press*, San Diego, 1999.