Article

Quantum Complexity

by Devang Bajpai

What is Complexity?

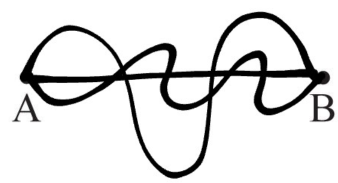

Fig: Shortest Distance between two points A & B Devang Bajpai

Let us start by asking a simple question- what is the absolute smallest path between A to B? Complexity is the absolute minimum number of steps needed to go from A to B. Does it significantly matter how many steps are involved? Generally speaking, it doesn't. Considering another example, let us discuss the complexity of classical states of N bits. Now, N bits refer to the binary units that are in a computer, i.e., $\uparrow$ and $\downarrow$.

$$\text{A simple state: } \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \ldots$$

I will take the simplest state to be the state in which all the bits are up. And here these are simple but universal gates. Now, a gate is simply an operation that you can do on these bits, and it involves only small number of bits. Gates can be assembled into quantum circuits in the manner shown in Fig. 6.1b. The simplest gate would be the 'flip gate' in which you simply flip one bit, and that's again, a simple but universal gate. The maximum number of gates that you will ever need to get the final output is no more than N. So the maximum classical state complexity will be N.

Feynman[1] pointed out that the quantum state can be more complex than the classical state. A quantum state can be thought of as linear superposition with coefficient ψ_i

$$|\psi\rangle = \sum_{\text{classical state}} \psi_i |classical\rangle$$

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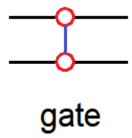
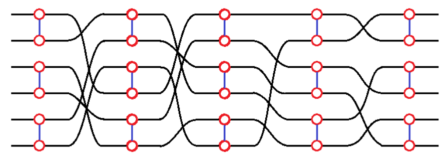
[1] R. P. Feynman, Simulating Physics with Computers. International Journal of Theoretical Physics, 21(6), 467–488, 1982.



(a) A gate acts on an incoming state of two-qubits to give an outgoing state. Leonard Susskind, arXiv:1810.11563v1



(b) Standard circuit architecture. Leonard Susskind, arXiv:1810.11563v1

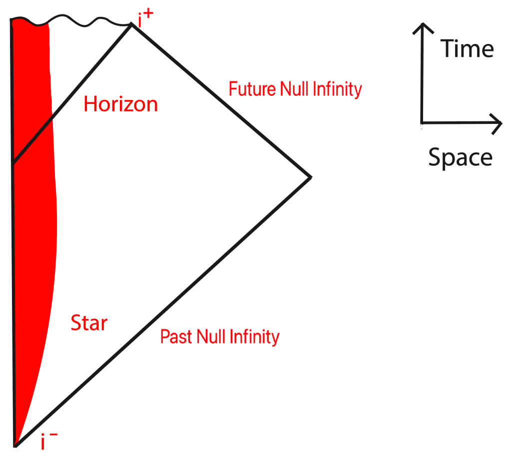
$$\psi_i = 2^N \text{complex Amplitude}$$

For N quantum bits there are 2^N classical states[2] and that is the reason that quantum states are computationally hard. Moreover, other than complexity of states, there are also the complexities of unitary operators and so forth, but I will not get into those.

The maximum quantum complexity that can ever happen is instead of being N, as it is for classical physics, it is e^{2N}, and almost all states have close to maximal complexity. Suppose you are given a quantum state, and you want to compute quantum complexity. To answer this question you can go back to the question of what is the minimal distance on geometry from one point to another. It is easy to find geodesics but it's very hard to know whether the geodesic you found is the minimal.

So while complexity is a well defined thing it can be relatively small, since it doesn't have to be very big. It is generally very hard to compute. The lesson from all this is that quantum complexity is a very peculiar quantity, and you can't tell the difference between a large value and a small value; it's ethereal. And for computer scientists it is almost imperceptible, but this makes it even more interesting- like energy, entropy,and density, simply by the virtue of property that they are imperceptible.

In 2014, *Leonard Susskind*[3] conjectured that the volume of the interior of a black hole is the complexity of it's quantum state. So for physicists, it's the interior of a black hole.



Penrose Diagram for a Black Hole Edward Witten, arXiv:2412.16795v1

Black Holes: Volume ∼ Complexity

Talking about black holes, according to Hawking and Bekenstein, the entropy term is given by the area of the black hole with a numerical factor,

$$S = \frac{A}{4G\hbar}.$$

The second law of thermodynamics says that entropy increases with time but until you get to the thermal equilibrium (and that takes a very short time after that black hole is completely static, at least from outside). If we plot the area and entropy we get a similar plot like the complexity graph Fig.6.2.

In a black hole, the volume of the interior increases for a very long time and it goes much like entropy. Now complexity grows for a long period of time until it hits the maximum possible complexity of a quantum system. For a solar mass black hole, the time that it takes to equilibrate is 10^{-3} seconds. Technically, the process is called the 'scrambling'.

In the Penrose diagram of a black hole, the complexity of the system continues to evolve and grow for an incredibly long time. This growth resembles the behavior of entropy but occurs on a vastly larger timescale—on the order of $\exp 10^{78}$ years for a solar mass black hole. During this immense duration, the black hole interior's complexity increases steadily until

[2] A Complex Amplitude describes the magnitude and phase of a wave or quantum state. It is represented as a complex number, where the real part corresponds to the wave's magnitude and the imaginary part encodes its phase.

[3] L. Susskind,"Computational Complexity and Black Hole Horizons," Fortsch. Phys. 64 (2016), 24-43 doi:10.1002/prop.201500092 [arXiv:1403.5695 [hep-th]

quantum fluctuations eventually accumulate and halt this growth. This phenomenon is depicted in figure 6.2.

Complexity Growth

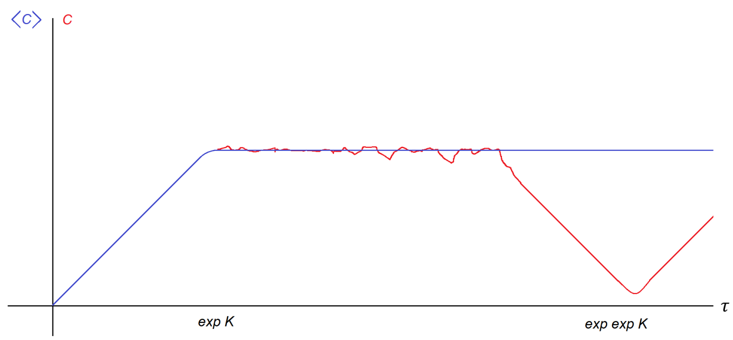


Figure 6.2: Evolution of complexity with time. The ragged red curve is the evolution of a specific instance of an ensemble. The smooth curve is the ensemble average. Leonard Susskind, arXiv:1810.11563v1

Now to determine how the complexity grows for some amount of time, the number of gates will almost continuously be the minimum number of gates that it takes to get where you're going, so it follows that complexity begins to grow until it hits the maximum possible complexity of a quantum system, which is exponential like entropy.

This looks very much like the second law of thermodynamics that is, the growth of complexity is governed by a principle analogous to the second law of thermodynamics, which we call the *second law of quantum complexity*.[4]

Concluding Remarks

In conclusion, quantum complexity is not merely a theoretical curiosity, it is a very important field of information theory and theoretical physics.[5] Its implications extend from advancing quantum computing to unraveling the mysteries of black holes and spacetime.

[4] A. R. Brown and L. Susskind, "Second law of quantum complexity," Phys. Rev. D 97 (2018) no.8, 086015 doi:10.1103/PhysRevD.97.086015 [arXiv:1701.01107 [hep-th]].

[5] See L. Susskind, "Three Lectures on Complexity and Black Holes," Springer, 2020, ISBN 978-3-030-45108-0, 978-3-030-45109-7 doi:10.1007/978-3-030-45109-7 [arXiv:1810.11563 [hep-th]]